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A novel method for reliability evaluation of Markov systems

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Abstract

This paper introduces a method based on the spectral representation of exponential and power operations to tackle the complexity issues associated with dependability characteristics in Markov systems. The conventional formulas become impractical due to the exponential growth in the number of states, which becomes a significant challenge in binary component systems. In such contexts, an approximation method is proposed in a controlled mathematical framework to realize accurate reliability assessment for highly reliable systems. In this paper, formulas for computing system dependability such as reliability, failure rate, conditional survival distribution, etc. are derived and proved to be useful and accurate for the reliability assessment of non-ergodic systems, while the availability of ergodic systems is derived. The only elements needed here are the largest eigenvalues and corresponding eigenvectors of the generate or subtransition matrix, which describes the approximation of reliability. Finally, the approximate results and exact results are verified through numerical examples, proving the correctness of the proposed method.

Keywords: Markov process, phase-type distribution, reliability, asymptotic failure rate, spectral analysis

1. Introduction

In practical engineering, the comprehensive advancement of technology and the diversification of demands have led to a continuous increase in the scale and complexity of systems. While this progression yields significant enhancements in performance, it has also heightened the demands on system reliability and security refer to [1]. Reliability analysis for complex systems including failure rate, availability, conditional reliability, and mean time to failure, has emerged as one of the prominent and challenging topics in academic research [2]. However, due to the inherent diversity and complexity of these systems, traditional reliability calculation methods become impractical when confronted with a large number of components. For instance, even a system composed of 20 binary components may exhibit over a million possible states, this necessitates managing matrices exceeding one million by one million dimensions. This renders traditional reliability calculation methods not only excessively complex but also computationally prohibitive refer to [3]. Therefore, it is crucial to obtaining approximate reliable results within a well-defined mathematical framework.

It is widely acknowledged that performing an exact reliability assessment is practically infeasible in most complex scenarios. This reality has spurred increasing interest in employing approximate methods for reliability analysis. The asymptotic approach is applicable as time t approaches infinity, offering significant simplification when the system or certain characteristics rapidly reach a steady-state regime refer to [4]. Alternatively, asymptotic analysis can also be performed in

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series or functional settings, particularly when considering a large family of systems under weak convergence schemes refer to [5]. Numerous researchers have conducted relevant studies, which can ultimately be classified into three main approaches. The first method involves aggregation techniques, which entail consolidating large subsets of states into a single macro-state, resulting in a more manageable system with fewer macro-states e.g. see [6–8]. However, it should be noted that typical reliability systems do not generally fulfill the exact aggregation conditions. Specifically, reversible Markov chains must possess a positive stationary distribution according to [9, 10], whereas irreversible Markov chains necessitate the application of more sophisticated methods refer to [11], which has brought challenges to the related research work. To address the limitations of this method, a second approach has emerged involving simplifications through asymptotic series schemes. The asymptotic method within this context is better suited to reliability problems, e.g. see [5, 12], primarily relying on the weak topology of stochastic process convergence within Skorohod space refer to [13]. Nevertheless, this methodology requires an extremely high level of mathematical foundation, which constitutes a considerable challenge for novices.

The spectral analysis method is employed to address these challenges and has found extensive application across various engineering domains, attracting growing attention from researchers in related fields [14, 15]. Traditionally, [16] explores the application of spectral analysis in digital transmission systems. In another study, [17] utilizes spectral analysis to extract signal features that enhance fault diagnosis techniques, thereby improving the accuracy and reliability of gear fault identification. Furthermore, spectral analysis can also be well applied in Markov systems. For instance, [18] describes the spectral algorithm for learning hidden Markov models and highlights that the relationship between observations and hidden states can often be uncovered using spectral methods correlating past and future observations. Further, [19] asserts that spectral analysis allows for an examination of the rank and characteristics of Markov processes, as well as their representability, aggregability, and generalizability. Nevertheless, the computational complexity of this method grows cubically with the number of system states, creating obstacles to its implementation in large-scale systems. Subsequently, [20] introduces the use of spectral representation in Markov switching bilinear processes and derives the corresponding spectral density function for the model. Meanwhile, [21] employs spectral analysis to investigate stability and performance indicators in queuing systems, such as waiting time distributions and service times. In recent years, [22] provides a systematic and comprehensive introduction to spectral graph theory applied in Markov chains, focusing on exploring properties of eigenvalues and eigenvectors corresponding to their transition matrices. The aforementioned findings indicate that the spectral analysis method offers significant theoretical support for the reliability assessment of system.

In engineering practice, system typically encompass multiple operational modes and states, as Markov processes can effectively characterize the dynamic transitions of system states through state transition probabilities. This approach has been extensively utilized across various domains, particularly in communication networks and financial models. Furthermore, the role of Markov systems in reliability evaluation has become increasingly prominent. By assessing potential system failures, these systems can more efficiently evaluate risks and facilitate the implementation of preventive measures. Specifically, some researchers aim to guide design decisions by establishing reliability assessment models for Markov systems that strike a balance between performance enhancement and cost efficiency, such as applying the model to a k-out-of-n structure within traffic system, refer to [23], and also marine power system, refer to [24], etc. Additionally, scholars have developed availability decision-making models for Markov systems to formulate more rational maintenance policies, providing valuable insights for practical engineering see, e.g. [25–27]. Although Markov systems have demonstrated significant advantages in the reliability assessment of many systems refer to [28, 29], it is notable that the state space of such systems is limited. When this method is applied to large-scale systems, the traditional exact solution methods show obvious deficiencies. However, thus far no studies have specifically employed approximation methods aimed at organizing the application of this approach concerning the reliability indicators of Markov systems.

To fill this void, this paper introduces asymptotic approximation methods for dependability, such as reliability, availability, asymptotic failure rate, conditional lifetime reliability and mean time to failure of Markov systems in continuous and discrete-time cases. This method only requires the Perron–Frobenius feature pair to represent the asymptotic reliability. Furthermore, some scholars have proposed another method, namely the free-eigenvector method. However, this necessitates obtaining all eigenvalues [see, e.g. 30]. It is widely accepted that employing spectral representation in computations requires calculating both the eigenvalues and eigenvectors of the underlying matrix. Consequently, we aim to simplify calculations by focusing solely on the largest eigenvalue from a portion of the generator (or from the transition matrix in discrete-time cases) corresponding to up states along with their respective right and left eigenvectors. These elements can be derived using specialized algorithms without needing to compute all eigenvalues or eigenvectors. This method yields highly accurate results, particularly for systems characterized by high reliability. The findings are illustrated through numerical examples. In summary, our contributions to existing theoretical research are outlined as follows:

- A novel method for assessing system reliability is proposed, which is based on spectral representation and is solved by calculating the eigenvalues and eigenvectors of the base matrix.
- The asymptotic method for the dependability analysis of Markov systems is investigated, which effectively solves the reliability assessment for homogeneous Markov systems.
- The dependability such as reliability, failure rate, and conditional survival distribution for both continuous and discrete time cases are approximated and compared with the exact results.

- Extended the model from non-ergodic systems to ergodic systems, and the asymptotic instantaneous availability formula is developed and verified with the steady-state availability.

The remainder of the paper is structured as follows: First, the non-ergodic system is considered, which is characterized by a single absorbing state. Based on this, section 2 provides detailed descriptions and fundamental assumptions regarding Markov systems, while developing approximate dependability measures such as reliability and failure rate, which are derived from continuous-time Markov processes of non-ergodic systems. Section 3 extends the approximation formulas to discrete-time cases of non-ergodic systems and proposes additional approximate formulas for reliability metrics. Subsequently, the model is extended from non-ergodic to ergodic cases. Section 4 elaborates on Markov processes for ergodic systems, providing approximate expressions for asymptotic instantaneous availability and steady-state availability functions. Section 5 provides numerical examples, with one related to continuous time and the other to discrete time, along with a detailed discussion on the ergodic case. Concluding remarks are also presented in section 5.

2. Research on the asymptotic for non-ergodic system

For non-periodic Markov systems, it is firstly examining the continuous-time case and subsequently transition to the discrete-time domain in the following subsection. In both scenarios, the system under consideration is non-ergodic and characterized by a single absorbing state. An absorbing state refers to a condition within a system from which escape is impossible once it has been entered. In practical applications, an absorbing state may represent a terminal condition, such as the completion of a chemical reaction or the attainment of equilibrium in a physical system. The presence of an absorbing state significantly influences both the behavior and long-term characteristics of the system.

2.1. Continuous-time case

For the convenience of the subsequent derivation, we first give the following definitions. Let $\mathbb{N}$ denotes the set of natural numbers $\{0, 1,\dots\}$, and $\mathbb{N}^*$ is defined as $\mathbb{N}^* \setminus \{0\}$. Let $\mathbb{R}_+ = [0, \infty)$ denote the set of positive real numbers thereafter.

2.1.1. Reliability and failure rate. To begin with, we first consider a random system whose temporal behavior is characterized by a continuous-time homogeneous Markov process $\{X_t, t \geqslant 0\}$ with a finite state space $E = \{1,2,\dots,r+1\}$ and generator matrix $\boldsymbol{A} = (a_{ij}, i,j \in E)$. The subset $E_0 = \{1,2,\dots,r\}$ comprises the working (up) states, whereas state $r+1$ represents a failure (down) state. Furthermore, define the lifetime of this system as T, given by

$$T := \inf\{t \geqslant 0 : X_t = r+1\}.$$

This expression represents the earliest time t at which the system transitions to state $r+1$. Generally, set $\inf\{\} = \infty$, but in this context, there is no need for such designation since state $r+1$ will inevitably be reached in a finite time with probability one.

Let us define the following partition of the generator $\boldsymbol{A}$, following the sets E_0 and $\{r+1\}$, $\boldsymbol{A}_0$ represents the up states, while $\boldsymbol{A}_1$ represents the down states, then we have

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 \\ \boldsymbol{0} & 0 \end{pmatrix},$$

and the initial row vector $\boldsymbol{\alpha}_0$ of the process X_t i.e. the law of the r.v. X_0,

$$\boldsymbol{\alpha} = (\boldsymbol{\alpha}_0, \alpha_{r+1}),$$

where $\boldsymbol{\alpha}_0$ is the initial vector on E_0, and then define the vector $\mathbf{1}$, an r-dimensional column vector of ones [see, e.g. 31, 32].

Subsequently, let us denote F, f and R as the distribution function, the probability density function and the reliability function of T, respectively. The distribution function of T is, see, i.e. [33]

$$1 - F(t) = R(t) = \boldsymbol{\alpha}_0 \mathrm{e}^{\boldsymbol{A}_0 t}\mathbf{1}, t \geqslant 0. \tag{1}$$

This is a continuous-time phase-type distribution [see, i.e. 34, 35].

It is worth noticing that in the following sections, we will represent probability vectors as row vectors and deterministic vector functions as column vectors. Consequently, we denote the standard failure rate of the system by $\lambda(t)$, which can be expressed as

$$\lambda(t) = \frac{f(t)}{R(t)} = -\frac{R'(t)}{R(t)}, t \geqslant 0. \tag{2}$$

Furthermore, we will present this general definition in the context of the specific case of the aforementioned Markov system.

Since reliability serves as a crucial indicator for the system and is challenging to compute for some complex systems, the asymptotic results of system reliability are presented below. This approximation closely aligns with the true value of reliability. In section 4, we will provide comparison curves that illustrate both the exact and approximate results. For this purpose, the following assumptions are given that matrix $\boldsymbol{A}_0$ is irreducible. In that case, there exists a real eigenvalue of $\boldsymbol{A}_0$, denoted as s_0, which is greater than the real part of any other eigenvalue, i.e. $s_0 > Re(s)$, s is any other eigenvalue of $\boldsymbol{A}_0$. Which is greater than the real part of any other eigenvalue, i.e. $s_0 > Re(s)$, s is any other eigenvalue of $\boldsymbol{A}_0$.

Proposition 2.1. *The asymptotic reliability is expressed as*

$$R(t) = K\mathrm{e}^{-|s_0|t} + o\left(\mathrm{e}^{-|s_0|t}\right),$$

where the constant $K := \boldsymbol{\alpha}_0 \boldsymbol{B}_0 \boldsymbol{1}$*, and the matrix* $\boldsymbol{B}_0 := uv^{\mathrm{T}}/u^{\mathrm{T}}v$*, with u and v the positive right and left eigenvectors of* $\boldsymbol{A}_0$ *corresponding to the largest eigenvalue* s_0*. The prime here,* u^{T}*, is the transpose of the column vector u, so* u^{T} *is a row vector.*

The vectors here are column vectors and $\boldsymbol{a}'$ means the transpose of vector $\boldsymbol{a}$. Let us start by a preliminary result. This result is well known, under different forms, in spectral analysis of exponential operators [see, e.g. 36, 37]. Based on this, a lemma is given as follows.

Lemma 2.1. *If states* E_0 *are a communication class, then we have the following representation*

$$\mathrm{e}^{\boldsymbol{A}_0 t} = B_0 \mathrm{e}^{-|s_0|t} + \boldsymbol{o}\left(\mathrm{e}^{-|s_0|t}\right), t \to \infty,$$

where $B_0 = uv^T/u^Tv$ is a square matrix $r \times r$ and $\boldsymbol{o}\left(\mathrm{e}^{-|s_0|t}\right)$ is a matrix small 'o' with elements of order $o\left(\mathrm{e}^{-|s_0|t}\right)$.

Proof. The proof of this result is obtained in a straightforward way by bringing lemma 2.1 into formula (1). □

This proves proposition 2.1.

On this basis, here also given that $\boldsymbol{A}_0$ is irreducible, the asymptotic expression of failure rate can be naturally obtained.

Proposition 2.2. *(i) The failure rate of a Markov system [32] is given as*

$$\lambda(t) = -\frac{\boldsymbol{\alpha}_0 \boldsymbol{A}_0 \mathrm{e}^{\boldsymbol{A}_0 t} \boldsymbol{1}}{\boldsymbol{\alpha}_0 \mathrm{e}^{\boldsymbol{A}_0 t} \boldsymbol{1}}.$$

(ii) The asymptotic failure rate is

$$\boldsymbol{\lambda}_\infty := \lim_{t\to\infty} \lambda(t) = -\frac{\boldsymbol{\alpha}_0 \boldsymbol{A}_0 \mathrm{e}^{\boldsymbol{A}_0 t} \boldsymbol{1}}{\boldsymbol{\alpha}_0 \mathrm{e}^{\boldsymbol{A}_0 t} \boldsymbol{1}}.$$

Proof. It can be directly derived by taking the derivative of the formula (1) with respect to t and then substituting it into the formula (2). □

For proposition 2.2(ii). From the above lemma 2.1, we can obtain

$$\boldsymbol{\alpha}_0 \mathrm{e}^{\mathbf{A}_0 t} 1 = k\mathrm{e}^{-|s_0|t} + o\left(\mathrm{e}^{-|s_0|t}\right), t \to \infty$$

where $K := \boldsymbol{\alpha}_0 \boldsymbol{B}_0 \mathbf{1}$.

Now, from the formula of $\lambda(t)$ in (i), we can write as follows when $t \to \infty$,

$$\begin{aligned}
\lambda(t) &= -\frac{\boldsymbol{\alpha}_0 \boldsymbol{A}_0 \left[\boldsymbol{B}_0 \mathrm{e}^{-|s_0|t} + o\left(\mathrm{e}^{-|s_0|t}\right)\right] \mathbf{1}}{K\mathrm{e}^{-|s_0|t} + o\left(\mathrm{e}^{-\lambda t}\right)} \\
&= -\frac{\boldsymbol{\alpha}_0 \boldsymbol{B}_0 \boldsymbol{A}_0 \mathbf{1} \mathrm{e}^{-|s_0|t} + o\left(\mathrm{e}^{-|s_0|t}\right)}{K\mathrm{e}^{-|s_0|t} + o\left(\mathrm{e}^{-|s_0|t}\right)} \\
&= \frac{K_1 \mathrm{e}^{-|s_0|t} + o\left(\mathrm{e}^{-|s_0|t}\right)}{K\mathrm{e}^{-|s_0|t} + o\left(\mathrm{e}^{-|s_0|t}\right)} \\
&= \frac{K_1 + o(1)}{K + o(1)} \\
&\to \frac{K_1}{K}, t \to \infty.
\end{aligned}$$

Note that we have $-K_1 = \boldsymbol{\alpha}_0 \boldsymbol{B}_0 \boldsymbol{A}_0 \mathbf{1}$ ($\boldsymbol{A}_0$ and $\boldsymbol{B}_0$ are commute), and K is as defined above. This proves proposition 2.2.

2.1.2. Conditional reliability.

In addition to reliability and failure rate, conditional reliability also plays a crucial role in ensuring system operation. It emphasized the minimum estimated remaining time before system failure, which can provide valuable decisions for system maintenance and minimize the risk of unplanned downtime refer to [38].

Consequently, the formal definition of the conditional survival distribution of the system at a fixed time $t \geqslant 0$ is presented. This pertains to the remaining lifetime at time t, namely, $T-t$, assuming that the system has not failed before t, then we have

$$R_t(x) := \mathbb{P}(T - t > x | T\rangle t) = \frac{R(t+x)}{R(t)}. \tag{3}$$

The following proposition has also been proved by [12], through Laplace transform and complex analysis. For better understanding, here we provide a simpler and more comprehensive proof based on lemma 2.1 above.

Proposition 2.3. *(i) The conditional reliability at* $t \geqslant 0$ *is given by*

$$R_t(x) = \frac{\boldsymbol{\alpha}_0 \boldsymbol{A}_0 (t+x) \mathbf{1}}{\boldsymbol{\alpha}_0 \mathrm{e}^{\boldsymbol{A}_0 t} \mathbf{1}}.$$

(ii) The asymptotic conditional reliability can be written as

$$R_\infty(x) := \lim_{t\to\infty} R_t(x) = \mathrm{e}^{-|s_0|x},$$

where $\boldsymbol{A}_0$ *is irreducible.*

Proof. For (i), it can be proved directly by formula (3) and proposition 2.1. □

For (ii), for any fixed $x > 0$ as $t \to \infty$, we can obtain

$$\begin{aligned}
R_t(x) &= -\frac{\boldsymbol{\alpha}_0 \left[\boldsymbol{B}_0 \mathrm{e}^{-|s_0|(t+x)} + o\left(\mathrm{e}^{-|s_0|t}\right)\right] \mathbf{1}}{K\mathrm{e}^{-|s_0|t} + o\left(\mathrm{e}^{-|s_0|t}\right)} \\
&= -\frac{K\mathrm{e}^{-|s_0|(t+x)} + o\left(\mathrm{e}^{-|s_0|t}\right)}{K\mathrm{e}^{-|s_0|t} + o\left(\mathrm{e}^{-|s_0|t}\right)} \\
&= \frac{K\mathrm{e}^{-|s_0|(t+x)} + o\left(\mathrm{e}^{-|s_0|t}\right)}{K + o(1)} \\
&= \frac{K\mathrm{e}^{-|s_0|x} + o(1)}{K + o(1)} \\
&\to \mathrm{e}^{-|s_0|x}, t \to \infty.
\end{aligned}$$

This proves proposition 2.3.

In reliability-related issues, $E(T)$ represents the mean time to failure, which is usually defined by MTTF. We can also give the conditional MTTF at time t as

$$\mathrm{MTTF}_t := E[T - t | T > t] = \int_0^\infty \frac{R(t+x)}{R(t)} \mathrm{d}x. \tag{4}$$

Proposition 2.4. *(i) The conditional MTTF at time t is*

$$\mathrm{MTTF}_t = -\frac{\boldsymbol{\alpha}_0 \boldsymbol{A}_0{}^{-1} \mathrm{e}^{\boldsymbol{A}_0 t}\mathbf{1}}{\boldsymbol{\alpha}_0 \mathrm{e}^{\boldsymbol{A}_0 t}\mathbf{1}}.$$

(ii) The asymptotic MTTF_t for $t \to \infty$ is given by

$$\mathrm{MTTF}_\infty := \lim_{t\to\infty} \mathrm{MTTF}_t = \frac{K_2}{K},$$

where $K_2 := -\boldsymbol{\alpha}_0 \boldsymbol{A}_0^{-1} \boldsymbol{B}_0 \mathbf{1}$ and $K := \boldsymbol{\alpha}_0 \boldsymbol{B}_0 \mathbf{1}$, as defined before, and $\boldsymbol{A}_0$ is irreducible.

Proof. For (i), It can be proved directly by formula (4) and proposition 2.1. □

For (ii), now for $t \to \infty$, proposition 2.4(i) and lemma 2.1 can be obtained as

$$\begin{aligned}
\mathrm{MTTF}_t &= -\frac{\boldsymbol{\alpha}_0 \boldsymbol{A}_0{}^{-1} \mathrm{e}^{\boldsymbol{A}_0 t}\mathbf{1}}{\boldsymbol{\alpha}_0 \mathrm{e}^{\boldsymbol{A}_0 t}\mathbf{1}} \\
&= -\frac{\boldsymbol{\alpha}_0 \boldsymbol{A}_0{}^{-1}\left[B_0 \mathrm{e}^{-|s_0|(t+x)} + \boldsymbol{o}\left(\mathrm{e}^{-|s_0|t}\right)\right]\mathbf{1}}{K\mathrm{e}^{-|s_0|t} + o\left(\mathrm{e}^{-|s_0|t}\right)} \\
&= \frac{K_2 \mathrm{e}^{-|s_0|t} + o\left(\mathrm{e}^{-|s_0|t}\right)}{\alpha_0 \mathrm{e}^{\boldsymbol{A}_0 t}\mathbf{1}} \\
&= \frac{K_2 + o(1)}{K + o(1)} \\
&= \frac{K_2}{K}, t \to \infty,
\end{aligned}$$

where $K_2 := \boldsymbol{\alpha}_0 \boldsymbol{A}_0^{-1} \boldsymbol{B}_0 \mathbf{1}$, and the proof is completed.

This proves proposition 2.4.

2.2. Discrete-time case

Compared to the preceding section, this section focuses on approximation methods within the discrete framework. Specifically, the system under consideration is characterized by a Markov chain $X_n, n \geqslant 0$, which operates within the state space $E = \{1,\ldots\ r,\ r+1\}$. The transition dynamics of this Markov chain are governed by the transition matrix $\boldsymbol{P} = (\boldsymbol{P}_{ij}, i, j \in E)$ and $\boldsymbol{P}_0$ the restriction of $\boldsymbol{P}$ on $E_0 \times E_0$. That is

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_0 & \mathbf{P}_1 \\ \mathbf{0} & 1 \end{pmatrix},$$

and the initial row vector $\boldsymbol{\alpha}$ of the chain X_n, i.e. the law of the r.v. X_0,

$$\boldsymbol{\alpha} = (\boldsymbol{\alpha}_0, \alpha_{r+1}),.$$

This proves proposition 2.4.

where $\boldsymbol{\alpha}_0$ is the initial probability vector on the states E_0. T is the lifetime defined as in the continuous-time case.

Let $p = (p_k, k \in \mathbb{N})$ be the law of T, i.e.

$$p_k := \mathbb{P}(T = k) = \boldsymbol{\alpha}_0 \boldsymbol{P}_0^{k-1}(I - \boldsymbol{P}_0)\mathbf{1}, k \geqslant 1,$$

and $p_0 = \alpha_{r+1}$, which is the well-known discrete-time Ph-distribution, for details refer to [34].

Therefore, employing the concept of reliability which indicates that no failure occurred before $k \in \mathbb{N}$ is, see, [39],

$$R(k) = \boldsymbol{\alpha}_0 \boldsymbol{P}_0^k \mathbf{1},$$

and $R(0) = 1$, $\alpha(t) = 0$.

Afterwards, the failure rate in discrete-time is defined by a sequence of positive real numbers $(\lambda_k,\ k \geqslant 0)$,

$$\lambda_k := \mathbb{P}(T = k | T \geqslant k) = 1 - \frac{R(k)}{R(k-1)}, k \geqslant 1,$$

and $\lambda_0 = 0$.

Since this failure rate is defined by [40], which is termed as BMP-failure rate. This is a probability and consequently it takes values into the real interval [0,1] which is different from the continuous-time case where the failure rate takes vales within the positive real numbers. For this reason, [41] introduced a new more adapted failure rate with values into the positive real numbers, and termed it as RG-failure rate, see also [39]. Then we have

$$r_k = \ln\frac{R(k-1)}{R(k)}, k \geqslant 1,$$

and $r_0 = 0$. In fact, $r_k = -\ln(1 - \lambda_k)$. It is worth noticing here that for small values of λ_k, we have that the two values λ_k and r_k are very close to each other.

The primary assumption here is that the matrix $\boldsymbol{P}_0$ is irreducible. Denote by q_0 the largest eigenvalue of $\boldsymbol{P}_0$, which is of multiplicity one, and u and v are the positive right and left corresponding eigenvectors. Based on these, some useful conclusions can be drawn as follows.

Proposition 2.5. *The asymptotic form of reliability is expressed as*

$$R(k) = \boldsymbol{D}_0 q_0^k + o\left(q_0^k\right), k \to \infty.$$

Proposition 2.6. *(i) The BMP-failure rate of the above system is*

$$\lambda_k := 1 - \frac{\boldsymbol{\alpha}_0 \boldsymbol{P}_0^k \mathbf{1}}{\boldsymbol{\alpha}_0 \boldsymbol{P}_0^{k-1} \mathbf{1}}, k \geqslant 1,$$

and $\lambda_0 = 0$.

(ii) The RG-failure rate is

$$r_k := \ln\frac{\boldsymbol{\alpha}_0 \boldsymbol{P}_0^{k-1} \mathbf{1}}{\boldsymbol{\alpha}_0 \boldsymbol{P}_0^k \mathbf{1}}, k \geqslant 1,$$

and $r_0 = 0$.

(iii) The asymptotic failure rates under the assumption of the irreducibility of $\boldsymbol{P}_0$ *is given as*

$$\lambda_\infty := \lim_{k\to\infty} \lambda_k = 1 - q_0,$$

and

$$r_\infty := \lim_{k\to\infty} r_k = -\ln q_0.$$

2.2.1. Conditional reliability. The logic for the conditional reliability definition is that at a fixed time $k \geqslant 0$, as in the continuous-time case, is denoted by

$$R_k(m) := \mathbb{P}(T-k>m|T>k) = \frac{R(k+m)}{R(k)},$$

for $k=0$, we have $R_0(m) = R(m)$. T is the system lifetime, Generally, set $\inf\{\} = \infty$ for the T, which ensures $T=\infty$ is well-defined in the theoretical case where the system never fails, even though T is finite with probability one in our irreducible Markov system.

Proposition 2.7. *(i) The conditional reliability at time* $k \geqslant 0$ *is*

$$R_k(m) = \frac{\boldsymbol{\alpha}_0\boldsymbol{P}_0^{k+m}\mathbf{1}}{\boldsymbol{\alpha}_0\boldsymbol{P}_0^{k}\mathbf{1}}.$$

(ii) The asymptotic conditional reliability, under the assumption that $\boldsymbol{P}_0$ *is irreducible, is*

$$R_\infty(m) := \lim_{k\to\infty} R_k(m) = q_0^m.$$

We can define also the conditional MTTF at time k

$$\mathrm{MTTF_k} := \mathrm{E}[T-k|T>k] = \sum_{m\geqslant 0}\frac{R(k+m)}{R(k)}.$$

Proposition 2.8. *(i) The conditional MTTF at time k is*

$$\mathrm{MTTF_k} = \frac{\boldsymbol{\alpha}_0\boldsymbol{P}_0^{k}(\boldsymbol{I}-P_0)\mathbf{1}}{\boldsymbol{\alpha}_0\boldsymbol{P}_0^{k}\mathbf{1}}.$$

(ii) The asymptotic $MTTF_\mathrm{k}$*, when* $k\to\infty$*, under the assumption that* $\boldsymbol{P}_0$ *is irreducible, is*

$$\mathrm{MTTF}_\infty := \lim_{k\to\infty} MTTF_k = 1 - q_0.$$

The proofs of propositions 2.5–2.8, concerning the discrete-time, are based on a similar result, that is lemma 2.2 below. This is also a well-known result, [see, e.g. 36, 37].

Let q_0 *be the greatest eigenvalue of the matrix* $\boldsymbol{P_0}$ *and u and v the corresponding right and left eigenvectors.*

Lemma 2.2. *If states* E_0 *are a transient communication class, then*

$$\boldsymbol{P}_0^k = \boldsymbol{C}_0 q_0^k + \boldsymbol{o}\left(q_0^k\right), k\in\mathbb{N},$$

where $\boldsymbol{C_0} := \boldsymbol{u}\boldsymbol{v}^\mathrm{T} / \boldsymbol{u}^\mathrm{T}\boldsymbol{v}$.

Then propositions 2.5–2.8 are proved in a very similar way as in the case of continuous-time.

3. Research on the asymptotic for ergodic system

In this section, we extend the model mentioned before from non-ergodic to ergodic. In engineering practice, ergodicity also seems as a measure of the stability of the system which can refer to [42, 43] for details. Let us consider an ergodic system with finite state space $E = \{1,\dots d\}$, $d \geqslant 2$, i.e. irreducible, which can be split into two subsets, say E_0 and E_1 described by a Markov process X. The states in E_0 are working states and those in E_1 are failure states [see, e.g. 33]. Without loss of generality, we assume that working states are first enumerated from 1 to r and then the down states are enumerated from $r+1$ to d.

To begin with, α is denoted as a row d-dimensional vector of the initial law of the Markov process. Furthermore, we also denote $\mathbf{1}_{d,r}$ as the d-dimensional column vector with one in the first r places and zeros in the others, then we have $\mathbf{1}_{r,r} = \mathbf{1}$.

The generator $\boldsymbol{A}$ of the Markov process has the eigenvalue 0, with multiplicity one. Let us consider the second largest eigenvalues of $\boldsymbol{A}$, say $s_1 < 0 = s_0$.

Denote also by $\boldsymbol{\pi}$ the stationary probability of the Markov process X, i.e. a row vector $\boldsymbol{\pi}$ of the stationary law which satisfies the equation $\boldsymbol{\pi}\boldsymbol{A} = \mathbf{0}$.

The instantaneous availability is given by, see, [33, 44],

$$A(t) = \boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{A}t}\mathbf{1}_{d,r}. \tag{5}$$

The asymptotic for steady-state availability of such a system is well-known, see, e.g. [33], to be

$$A_\infty = \lim_{t\to\infty} A(t) = \boldsymbol{\pi}\,\mathbf{1}_{d,r}.$$

Proposition 3.1. *For the continuous-time case, the instantaneous availability* $A[t]$ $(t\in\mathbb{R}_+)$ *of the system has the following asymptotic representation*

$$A(t) = A_\infty + C_1\mathrm{e}^{-|s_1|t} + \boldsymbol{o}\left(\mathrm{e}^{-|s_1|t}\right), t\to\infty,$$

where the constant $C_1 = \boldsymbol{\alpha}\boldsymbol{B}_1\mathbf{1}_{\mathrm{d,r}}$ *and the square matrix* $\boldsymbol{B}_1 := \boldsymbol{u}_1{\boldsymbol{v}_1}^T / {\boldsymbol{u}_1}^T\,\boldsymbol{v}_1$*. The vectors* $\boldsymbol{u}_1$*,* $\boldsymbol{v}_1$ *are, respectively, the right and left positive eigenvectors of* $\boldsymbol{A}$*, corresponding to the eigenvalue* s_1*.*

For the discrete-time case, we have a similar representation of the system described by a Markov chain X_n. Instead of the generator $\boldsymbol{A}$ we have the transition matrix $\boldsymbol{P}$. Let the stationary law given by a row vector $\boldsymbol{\pi}$, i.e. $\boldsymbol{\pi}\boldsymbol{P} = \boldsymbol{\pi}$. Consider also the second largest eigenvalue $q1$ of $\boldsymbol{P}$, say $0 < q_1 < q_0 = 1$. Then the instantaneous availability is given by previous work [33, 44],

$$A(k) = \boldsymbol{\alpha}\boldsymbol{P}^k\mathbf{1}_{d,r}.$$

The asymptotic availability is given by the same formula as above in the continuous-time case, i.e. $A_\infty = \boldsymbol{\pi}\,\mathbf{1}_{d,r} = \sum_{i=1}^{r}\pi_i$.

Let us denote Π the matrix $\mathbf{1}\otimes\boldsymbol{\pi}$, i.e. the matrix $d\times d$, where each line is the stationary probability vector $\boldsymbol{\pi}$. It is worth noticing that we use the same notation $\boldsymbol{\pi}$ and Π for the continuous and discrete-time cases.

Lemma 3.1. *(i) (Continuous-time case). The transition probability matrix of the Markov process has the asymptotic representation*

$$P(t) - \Pi = \exp(\boldsymbol{A}t) - \Pi = \mathrm{e}^{-|s_1|t}B_1 + \boldsymbol{o}\left(\mathrm{e}^{-|s_1|t}\right).$$

(ii) (Discrete-time case). The transition probability matrix for k steps of time, of the Markov chain, has the asymptotic representation

$$P^k - \Pi = q_1^k B_1 + \boldsymbol{o}\left(q_1^k\right).$$

Proof. From the above formula of instantaneous availability (5), and lemma 3.1 (i), we have

$$\begin{aligned} A(t) &= \boldsymbol{\alpha}\left[\Pi + \mathrm{e}^{-|s_1|t}B_1 + \boldsymbol{o}\left(\mathrm{e}^{-|s_1|t}\right)\right]\mathbf{1}_{d,r} \\ &= A_\infty + C_1\mathrm{e}^{-|s_1|t} + o\left(\mathrm{e}^{-|s_1|t}\right). \end{aligned}$$

□

This proves proposition 3.1.

Proposition 3.2. *The instantaneous availability, $A(k)$, $k \in \mathbb{N}$, of the system has the following asymptotic representation*

$$A(k) = A_\infty + D_1 q_1^k + \boldsymbol{o}\left(q_1^k\right), k \to \infty,$$

where the constant $D_1 := \boldsymbol{\alpha}\boldsymbol{B}_1\boldsymbol{1}_{\mathrm{d,r}}$ and the square matrix $B_1 := \boldsymbol{u}_1\boldsymbol{v}_1{}^T / \boldsymbol{u}_1{}^T\, \boldsymbol{v}_1$. The vectors $\boldsymbol{u}_1$, $\boldsymbol{v}_1$ are, respectively, the right and left eigenvectors of $\boldsymbol{P}$, corresponding to the eigenvalue q_1.

Moreover, if we consider the ergodic process only up to the lifetime T, i.e. $T = \inf\{t \geqslant 0 : X_t \in E_1\}$, then this system is equivalent to the nonergodic system considered in the first part of this paper. That is the down states E_1 are replaced by one aggregated state, say Δ, instead of state $r + 1$, and the generator L of an equivalent process to X_t, up to the time T, say Y_t, $0 \leqslant t < T$, can be written as

$$L = \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_{01}\boldsymbol{1} \\ \boldsymbol{0} & 0 \end{pmatrix}.$$

Of course, up to the time T^-, we have $X_t(\omega) = Y_t(\omega)$, for all ω.

The proof of proposition 3.2 is similar to the previous one in continuous-time.

4. Numerical examples

As an illustration, the model proposed in this article can be widely used in many practical engineering scenarios, especially the reliability of dependability and survival analysis. In order to validate the utility of the proposed model, it is necessary to assess its effectiveness and practicality in real-world scenarios through rigorous testing and evaluation.

Example 1. (continuous-time non-ergodic system). (Continuous-time non-ergodic system). Specifically, take the rotor system of the aircraft engine as an example, which is the key rotating component of the aircraft engine, different tasks require various speeds, such as take-off, climb, cruise, and other working conditions. The rotation speed of the rotor system also varies according to different working conditions, which can be assumed as a Continuous Time Markov Process model. When the rotation speed is too high or even exceeds a certain value, it will cause dry friction between the rotor and the stator, directly leading to system failure, which can be viewed as entering an absorbing state. We assume that the aircraft will go through four states from being put into operation to failure, namely states 1, 2, 3, and 4. Then the state transition situation can be represented by a 4×4 matrix.

Furthermore, we assume a new rotor system is placed into service in perfect operating order at time $t = 0$. The working conditions that vary with time are represented by a Continuous-time Markov Process, denoted as Z_t. Specifically, its infinitesimal generator $\boldsymbol{A}$ can be assumed to be as follows

$$A = \begin{pmatrix} -0.6 & 0.3 & 0.2 & 0.1 \\ 0.1 & -0.7 & 0.4 & 0.2 \\ 0.1 & 0.4 & -0.8 & 0.3 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where states 1, 2, and 3 are transient which refers to environments where systems can function, and state 4 is absorbing which refers to the extremely harsh working conditions where the system is failed. The initial distribution vector is assumed to be $\boldsymbol{\alpha} = (1, 0, 0, 0)$, indicating that the system begins operating with the normal working condition. Since the maximum eigenvalue plays a critical role in the asymptotic method proposed above, and there are several algorithms to provide the eigenpair with the largest eigenvalue, see, e.g. *Power iteration*, with linear convergence, and *Gram iteration* with super-linear convergence [45]. In this article, the power iteration method is applied, and the maximum eigenvalue calculated by the power iteration method is the same as the traditional method, and meantime, when the eigenvalues tend to a steady state and finally converge to the maximum eigenvalue s_0, the calculating residual is less than 10^{-5}, which illustrates the correctness and accuracy of the method.

After that, based on the maximum eigenvalue we obtained, the dependability indexes asymptotic approximation results for Markov systems can be obtained respectively.

Based on proposition 2.1, the curve of the reliability variation is shown in figure 1. Besides, when $t \to \infty$, the reliability gradually approaches 0.

Derived from proposition 2.2, the curve of the failure rate variation can be drawn in figure 2. Besides, when $t \to \infty$, the failure rate can be calculated $\lambda_\infty = 0.215\,789$.

Based on proposition 2.3, the conditional survival distribution, or equivalently the conditional reliability, can be calculated as follows. The curve of conditional reliability $R_t(x)$ when $m = 1$, is shown in figure 3, the conditional reliability is $R_\infty = 0.805\,902$ when $t \to \infty$, while when $m = 6$ is shown in figure 4, the conditional survival time is $R_\infty = 0.273\,964$ when $t \to \infty$.

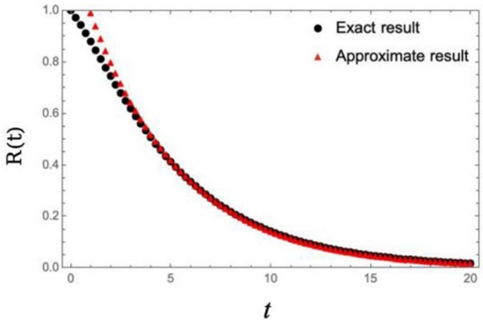

Figure 1. Curves of reliability $R(t)$.

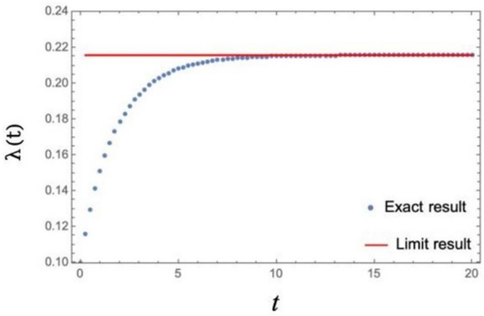

Figure 2. Curves of the failure rate $\lambda(t)$.

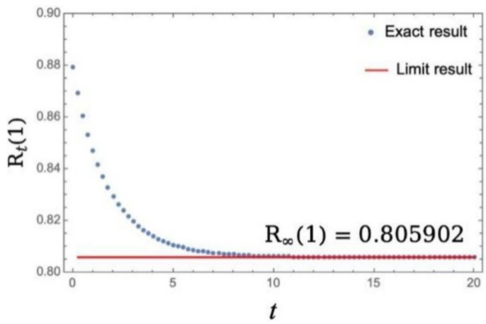

Figure 3. Curves of $Rt(x)$ when $x = 1$.

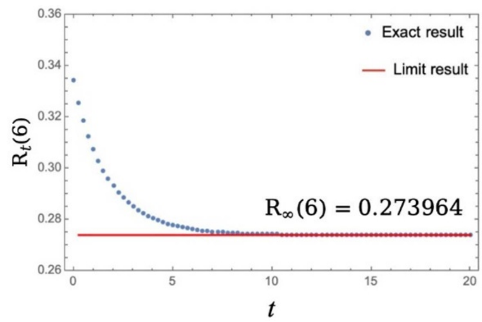

Figure 4. Curves of $Rt(x)$ when $x = 6$.

As we can know from the figure above the conditional survival time curve holds great significance in engineering and biomedical practice.

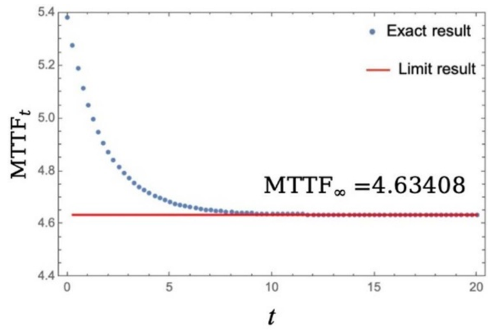

Figure 5. Curves of the MTTFt.

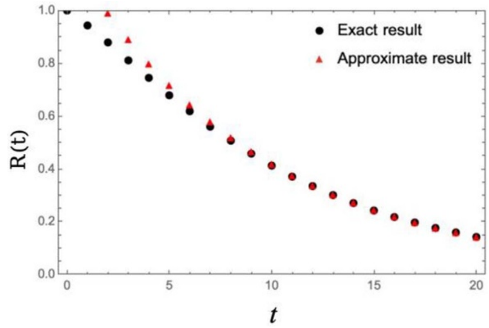

Figure 6. Curves of the $R(t)$.

Based on proposition 2.4, the conditional MTTF_t and $R(t)$ for the system can be drawn in figures 5 and 6. The asymptotic MTTF_t is $\mathrm{MTTF}_\infty = 4.63\,408$ when $t \to \infty$.

As indicated by the above results, the remaining reliability and conditional MTTF can be calculated at any given time for the system, and further provide valuable guidance for maintenance strategies.

Example 2. (Discrete-time non-ergodic system). Furthermore, we extend the continuous model to the discrete model by discretizing time t. The discrete model will be suitable for a wider range of scenarios, which is more convenient for calculation, and when the state space is large, discretization is more conducive to the convergence of the algorithm. Here a small $h > 0$ is taken as the step size and n is the step number, then $t = n{\cdot}h$. In that case, we have $\boldsymbol{P} = \boldsymbol{P}_h = \mathrm{e}^{\boldsymbol{A}h}$. Firstly, we assume $h = 0.5$.

According to proposition 2.6, the curve of the BMP-failure rate variation and the RG-failure rate variation for the above system are shown in figures 7 and 8.

It can be found that the results obtained by the two methods of calculating the failure rate are consistent with the results obtained by the asymptotic method, which verifies the effectiveness of the method when $k \to \infty$. In parallel, the failure rate attained by two methods is $\lambda_\infty = 0.102\,279$ and $r_\infty = 0.107896$, which are very close to each other, well-validated the correctness of the results.

Based on proposition 2.7, we can obtain the conditional reliability of the system at any time. When $m = 2$, the curve of the $R_k(m)$ variation is shown in figure 9, and the conditional

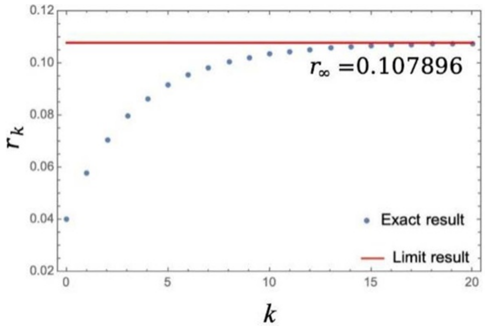

Figure 7. Curves of BMP-failure rate λ_k.

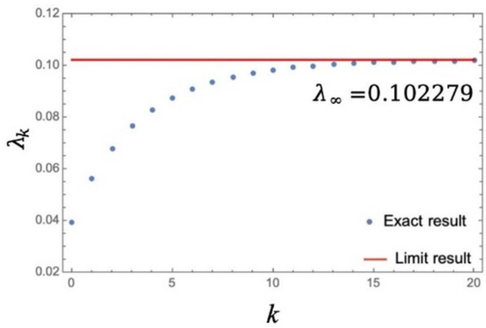

Figure 8. Curves of RG-failure rate r_k.

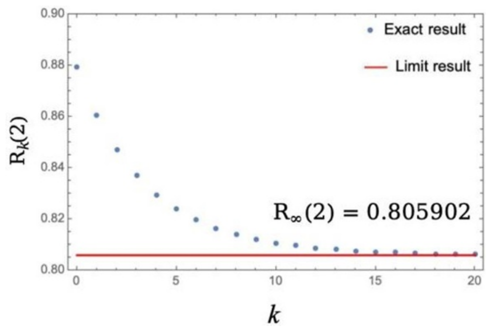

Figure 9. Curves of $R_k(m)$ when $m = 2$.

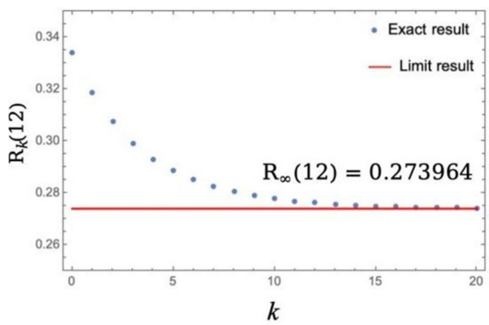

Figure 10. Curves of $R_k(m)$ when $m = 12$.

reliability can be calculated $R_k(2) = 0.805902$ when $k \to \infty$; the curve of conditional reliability when $m = 12$ is shown in figure 10, and the conditional reliability can be calculated $R_k(12) = 0.273964$ when $k \to \infty$.

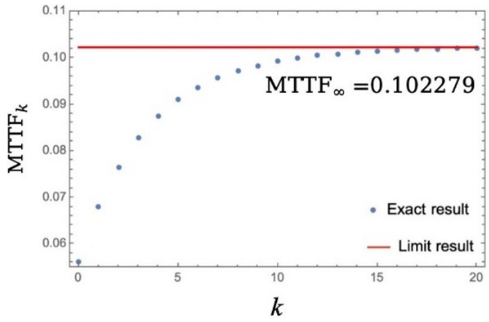

Figure 11. Curves of the MTTF_k.

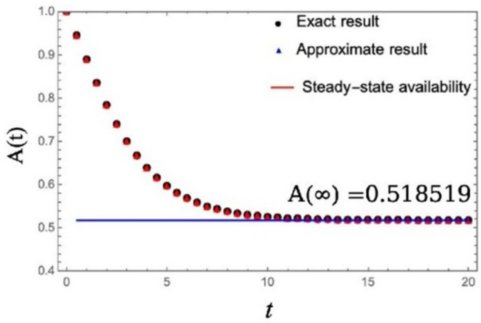

Figure 12. Curves of the $A(t)$ of ergodic Markov process.

Based on proposition 2.8, the curve of the MTTF_k is shown in figure 11. From the figure, we can see that when $k \to \infty$, the condition MTTF_k is 0.805 902.

Example 3. (Continuous-time ergodic system). In this case, the model is extended from non-ergodic system to ergodic system, which is also a very common situation in engineering practice. Then consider a modification of Example 1. It is assumed that the generator $\boldsymbol{A}'$is given by

$$\boldsymbol{A}' = \begin{pmatrix} -0.6 & 0.3 & 0.2 & 0.1 \\ 0.1 & -0.7 & 0.4 & 0.2 \\ 0.1 & 0.4 & -0.8 & 0.3 \\ 0.2 & 0 & 0 & -0.2 \end{pmatrix},$$

which is an ergodic Markov process with the same working and failed states. Based on proposition 3.1, the availability curves of the $A(t)$ are shown in figure 12.

It can be seen from figure 12 that the curves of instantaneous availability and steady-state availability almost completely overlap, and when $t \to \infty$, they are approximately equal to the value of the limit availability $A(t) = 0.518\,519$, which verifies the effectiveness and correctness of the proposed method, and also provides valuable suggestion for engineering practice.

Example 4. (Moran's reservoir system). To elucidate our model more effectively, here we introduce Moran's Reservoir model, which is extension of previous work [46]. Rooted in probability theory and Markov chain methods, this model is specifically designed to characterize the dynamic evolution of water volume in a finite-capacity reservoir under stochastic inflow and deterministic outflow conditions.

Specifically, we consider a reservoir with a fixed capacity of $c \in N^*$ units of volume, which is observed at discrete integer time points n. During each time interval $[n, n+1]$, a random quantity of water, denoted as Z_n units of volume, flows into the reservoir; notably, the inflow sequence Z_n is typically assumed to be serially independent. A core constraint of the model is the overflow mechanism: when the total water volume in the reservoir exceeds its capacity c after the inflow of Z_n, any surplus water overflows and is permanently lost. At the end of $[n, n+1]$, one unit volume of water, if available, leaves the reservoir. Let Xn denote the level of water in the reservoir at time n, thus, the level at time $n+1$ can be expressed as follows

$$X_{n+1} = (X_n + Z_n - 1)^+ \wedge (c-1)\,.$$

If the variables Z_n are assumed to be nonnegative and i.i.d. with common distribution $P(Z_n = k) = p_k$, then X_n is a Markov chain with state space $E = \{0, 1, \ldots, c-1\}$, and transition matrix for the system is

$$\begin{array}{c c}
 & \begin{array}{cccccc} 0 & 1 & 2 & \cdots & c-2 & c-1 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ c-2 \\ c-1 \end{array} &
\begin{pmatrix}
p_0+p_1 & p_2 & p_3 & \cdots & p_{c-1} & h_c \\
p_0 & p_1 & p_2 & \cdots & p_{c-2} & h_{c-1} \\
0 & p_0 & p_1 & \cdots & p_{c-3} & h_{c-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & p_1 & h_2 \\
0 & 0 & 0 & \cdots & p_0 & h_1
\end{pmatrix}
\end{array}$$

where $h_k = \sum_{i \geqslant k} p_i$ for $1 \leq k \leq c$ Note that c could be considered as a state of the chain, but then only X_0 could take this value. Therefore, to validate that our proposed model is also applicable to large-scale systems, we first assume $c = 1000$, resulting in a $c \times c$ matrix, with the other parameters set as follows: $p_0 = 0.655$; $p_1 = 0.05$; $p_2 = 0.05$; $p_3 = 0.145$; $p_4 = 0.08$; $p_5 = 0.02$. Based on these parameters, the availability is determined using proposition 4.1. Furthermore, P_0 for the system is defined as follows.

$$\begin{pmatrix}
p_1 & p_2 & \cdots & p_{c-2} & h_{c-1} \\
p_0 & p_1 & \cdots & p_{c-3} & h_{c-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & p_1 & h_2 \\
0 & 0 & \cdots & p_0 & h_1
\end{pmatrix},$$

what we consider as the lifetime of the reservoir system is the time to be empty, i.e, $T = \inf\{n \geqslant 1 : X_n = 0\}$. We suppose that $X_0 > 0$, so the reliability of time $n \geqslant 0$ is $R(n) = P(T > n)$.

The reliability for the reservoir system is calculated by proposition 3.1. Additionally, the comparison curve between the exact results and approximate results is presented below. Analysis of figures 13 and 14 reveal that when the system comprises 1000 states, both the exact and approximate results for reliability and availability are identical, thereby confirming the validity of the proposed method in large-scale systems.

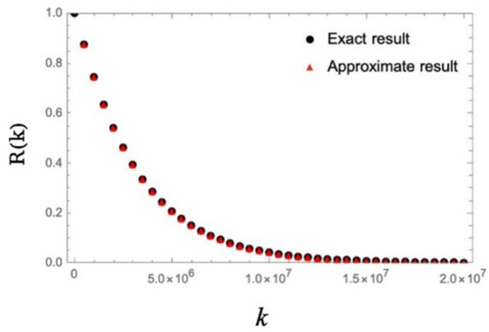

Figure 13. Curves of reliability $R(k)$ for reservoir system.

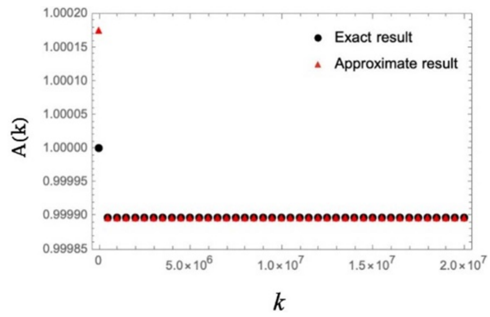

Figure 14. Curves of availability $A(k)$ reservoir system.

5. Concluding remarks

This paper introduces a streamlined computational method for assessing the dependability characteristics of large systems through an asymptotic approach. This methodology is predicated solely on the largest eigenvalue (the Perron–Frobenius eigenvalue) and the corresponding right and left eigenvectors of either the generator or transition matrix, along with the second largest eigenvalue pertaining to availability. In this article, we employ the power iteration method, which allows us to efficiently estimate the dependability characteristics of real systems modeled by a Markov process in either continuous or discrete time.

Despite the fact that the theoretical results are given for $t \to \infty$, we can see in the numerical examples, see e.g. Figure 15, that the approximate value coincides from the first few time units to the exact values. This is due to the very fast convergence of order $o(\mathrm{e}^{-|s_0|t})$.

What is remarkable here is that the performance degradation of the system is connected to the spectral properties of the generator or of the transition matrix. For example, propositions 2.1 and 3.1, indicate that reliability degradation is as $o(\mathrm{e}^{-|s_0|t})$ (or $o((q_0)^k)$. in the discrete-time case), as the time $t \to \infty$ (or $k \to \infty$), for continuous and discrete-time cases respectively.

For a very reliable system the largest eigenvalue $s_0 < 0$ ($q_0 \in (0,1)$ is very close to 1) is very close to zero and its degradation rate is slow.

The rate of convergence in the asymptotic representations depends in fact on the gap of the largest eigenvalue of the first and second eigenvalue, for the reliability characteristics, and from second and third eigenvalues, for availability, both in

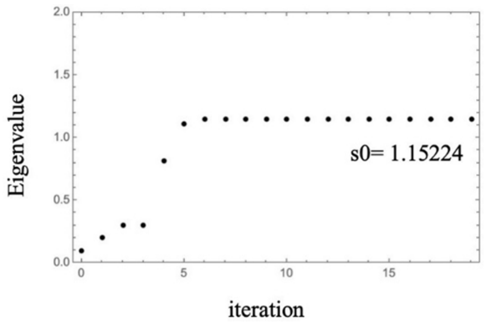

Figure 15. Curves of the maximum eigenvalue s_0.

continuous and discrete-time. That is, for the continuous-time case, for the reliability, $O(\mathrm{e}^{-(|s_1|-|s_0|)t})$, and for availability, $O(\mathrm{e}^{-(|s_2|-|s_1|)t})$, (big '$O$' here). For the discrete-time case we have also, for the reliability, $O((q_0-q_1)^k)$, and for availability, $O((q_1-q_2)^k)$. It is worth noticing here that $|s_1|-|s_0|>0$, $|s_2|-|s_1|>0$ and $0<q_0-q_1<1$, and $0<q_1-q_2<1$. It is worth noticing that in the non-ergodic case, the reliability is equal to availability. The results presented here can be generalized for the case of infinite countable state spaces.

In general, this paper proposes the derivation and application of the asymptotic method in simplifying the calculation of dependability characteristics of large-scale systems and verifies it in both ergodic and non-ergodic systems. Meantime, the power iteration method is used to directly calculate the system maximum eigenvalue. In future research, we will explore the application of more asymptotic methods in large-scale system reliability assessment, and explore more algorithms for quickly solving the maximum eigenvalue to provide meaningful reference for engineering practice.

Acknowledgments

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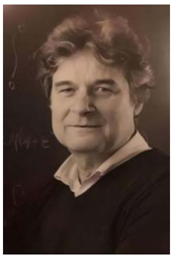

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